

# Pauli-Villars Regularization and Light Front Hamiltonian in (2+1)-dimensional Yang-Mills Theory

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## Abstract

The renormalization problem of (2+1)-dimensional Yang-Mills theory quantized on the light front is considered. Extra fields analogous to those used in Pauli-Villars regularization are introduced to restore perturbative equivalence between such quantized theory and conventional formulation in Lorentz coordinates. These fields also provide necessary ultraviolet regularization to the theory. Obtained results allow to construct renormalized Hamiltonian of the theory on the light front.

**Key words:** Pauli-Villars regularization, quantization on the light front, Yang-Mills theory.

## 1 Introduction

The present paper is devoted to quantization of field theory on the light front. Yu.V. Novozhilov was interested in this subject for many years, and he worked in this direction together with the part of authors of this paper.

Quantization of field theory on the light front (LF) [1, 2] allows to simplify the description of the vacuum state. This makes the application of nonperturbative Hamiltonian approach to the bound state and mass spectrum problem more convenient [2, 3]. The LF can be defined by the equation  $x^+ = 0$  where  $x^+ = \frac{x^0 + x^1}{\sqrt{2}}$  plays the role of time ( $x^0, x^1, x^\perp$  are Lorentz coordinates with  $x^\perp$  denoting the remaining spatial coordinates). The role of usual space coordinates is played by the LF coordinates  $x^- = \frac{x^0 - x^1}{\sqrt{2}}, x^\perp$ .

The generator  $P_-$  of translations in  $x^-$  is kinematical [1] (i.e. it is independent of the interaction and quadratic in fields, as a momentum in a free theory). On the other side it is nonnegative ( $P_- \geq 0$ ) for quantum states with nonnegative mass squared. So the state with the minimal eigenvalue  $p_- = 0$  of the momentum operator  $P_-$  can describe (in the case of the absence of the massless particles) the vacuum state, and it is also the state minimizing  $P_+$  in Lorentz invariant theory. This means that the physical vacuum turns

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out to coincide with the mathematical one. It is possible to introduce the Fock space on this vacuum and formulate in this space the eigenvalue problem for the operator  $P_+$  (i.e. the LF Hamiltonian). In this way one can find the spectrum of mass  $m$  in the subspace with fixed values of the momenta  $p_-, p_\perp$  [3]:

$$P_+|p_-, p_\perp\rangle = \frac{m^2 + p_\perp^2}{2p_-}|p_-, p_\perp\rangle, \quad (1)$$

see details in the review paper [4].

The theory on the LF has the singularity at  $p_- = 0$  [4]. To regularize it we use the cutoff  $p_- \geq \varepsilon > 0$ . In the present paper we consider a way to construct the perturbatively renormalized Hamiltonian on the LF (on difficulties to solve this problem see [5–11]). We consider this problem for (2+1)-dimensional Yang-Mills theory quantized on the LF where the above mentioned regularization  $|p_-| \geq \varepsilon > 0$  is applied. Unlike the early considered case of (3+1)-dimensional QCD [12], for (2+1)-dimensional model we can construct renormalized LF Hamiltonian containing no new unknown renormalization parameters.

It was shown in papers [4, 13, 14] that some diagrams of the perturbation theory, generated by the LF Hamiltonian, and corresponding diagrams of the conventional perturbation theory in Lorentz coordinates can differ. It was found that one can overcome this difficulty by addition of new (in particular, nonlocal) terms to the canonical LF Hamiltonian [12, 15, 16]. In the case of gauge field theory the infinite number of such terms appears [4, 12]. However one can avoid the differences between diagrams, generated by the LF Hamiltonian, and corresponding diagrams of the conventional perturbation theory if one adds extra ghost fields analogous to that in the Pauli-Villars (PV) regularization [4, 12]. In this way one can avoid the infinite number of terms mentioned above. It was also shown how to renormalize this theory ensuring the perturbative equivalence with the dimensionally regularized theory in Lorentz coordinates and the restoration of gauge invariance. For example, in (3+1)-dimensional Quantum Chromodynamics [12] this can require the inclusion of ten counterterms, necessary for UV renormalization, into LF Hamiltonian. It was shown that there must be the values of coefficients before these counterterms at which the restoration of gauge invariance occurs. However one cannot find these coefficients explicitly because of infinite number of divergent diagrams in (3+1)-dimensional theory. So one has to consider them as new unknown parameters.

The (2+1)-dimensional Yang-Mills theory, considered in the present paper, is super-renormalizable, so that all renormalizing counterterms can be found exactly via calculation of the finite number of diagrams. This allows to carry out the renormalization of the theory on the LF in such a way that no unknown quantities, besides the original parameters, appear. We regularize perturbative infrared (IR) divergences introducing topological Chern-Simons (CS) term [17, 18].

We analyse the perturbation theory, generated by quantization on the LF, and investigate its equivalence to the usual covariant perturbation theory in Lorentz coordinates. To do this we apply the method of paper [12]. We use the analog of the PV regularization to remove both UV divergences and differences between diagrams of perturbation theory on the LF and corresponding diagrams of covariant perturbation theory in Lorentz coordinates. We show how to restore gauge symmetry in the limit that removes PV regularization at the correct renormalization of the theory. In this way we can construct renormalized

Hamiltonian on the LF which can be used for nonperturbative calculation of mass spectrum in accordance with (1).

## 2 Divergences of (2+1)-dimensional Yang-Mills theory with Chern-Simons term

To construct the renormalized LF Hamiltonian we have to analyse diagrams of perturbation theory. These diagrams must be well defined, i.e. to be free of divergences. Yang-Mills theory in (2+1)-dimensions contains, besides usual UV divergences, also the IR ones [17,18]. That makes impossible the analysis of perturbation theory. As a solution to this problem we introduce the CS term [17,18], generating gluon field mass. In result we investigate the theory with the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \frac{m}{2}\varepsilon^{\mu\nu\alpha} \left( A_\mu^a \partial_\nu A_\alpha^a + \frac{2}{3}g f^{abc} A_\mu^a A_\nu^b A_\alpha^c \right), \quad (2)$$

where  $A_\mu^a(x)$  are gluon fields corresponding to gauge symmetry group  $SU(N)$ ,  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c$ ,  $a = 1, \dots, N^2 - 1$  are indices of adjoint representation,  $m$  and  $g$  are parameters,  $\varepsilon^{\mu\nu\alpha}$  is Levy-Civita symbol.

For the construction of LF Hamiltonian we take the gauge  $A_- = A^+ = 0$ . Its use in the action leads to the Lagrangian density in which the contribution from the first term in (2), having power four in fields, and the contribution from the CS term, having power three in fields, disappear:

$$\mathcal{L} = -\frac{1}{4}f^{a\mu\nu} f_{\mu\nu}^a + g f^{abc} A_+^a A_\perp^b \partial^+ A^{c\perp} + \frac{m}{2}\varepsilon^{\mu\nu\alpha} A_\mu^a \partial_\nu A_\alpha^a, \quad (3)$$

where  $f_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ . As a result, the propagator, in which the remaining part of CS term contributes, takes in the momentum space the following form:

$$\Delta_{\mu\nu}^{ab} = \frac{-i\delta^{ab}}{k^2 - m^2 + i0} \left( g_{\mu\nu} - \frac{k_\mu n_\nu + n_\mu k_\nu + i m \varepsilon_{\mu\nu\alpha} n^\alpha}{k_\parallel^2 + i0} 2k_+ \right), \quad (4)$$

where  $k_\parallel^2 = 2k_+ k_-$  and  $n_\nu$  is lightlike vector with components  $n_+ = 1$ ,  $n_- = n_\perp = 0$ . As one can see, the parameter  $m$  plays the role of the field  $A_\mu$  mass. For the regularization of singularity at  $k_- = 0$  in the propagator (4) the Mandelstam-Leibbrandt prescription is used [19,20]. Such a prescription allows to do the Wick rotation to Euclidean momentum space for diagrams where it is possible to analyse UV divergences of Feynman diagrams in the standard way.

The interaction term in the equation (3) leads to the vertex which contains derivative with upper index  $+$ . Let us remark that due to the global  $SU(N)$  symmetry (note that local, i.e. gauge symmetry, is broken by UV regularization in our approach) all diagrams with single external line must be equal to zero as they are vectors in the color space.

Let us find all UV divergent Feynman diagrams that must be renormalized. To do this we use the standard method of estimation of the UV divergency index of Feynman integrals in Euclidean space. In result the divergent diagrams are those shown in Fig. 1. As expected, their number is finite. Due to violation of Lorentz invariance by the introduction

of the  $A_- = 0$  gauge the number of divergent diagrams can increase, in principle. To check this it is necessary to analyse not only the total divergency index (in all components of the momentum) but also the UV divergency indices corresponding to only some part of components of the momentum. Taking into account the structure of the propagator (4) and the vertex one can see that only the divergency index in transverse component  $k_\perp$  can, in principle, exceed the total divergency index. However it is not difficult to verify that this case, in fact, does not realize, and the diagrams, shown in Fig. 1, exhaust all cases of the UV divergency. With respect to the UV divergency index the diagram in Fig. 1(a) is linearly divergent and the other diagrams are logarithmically divergent.

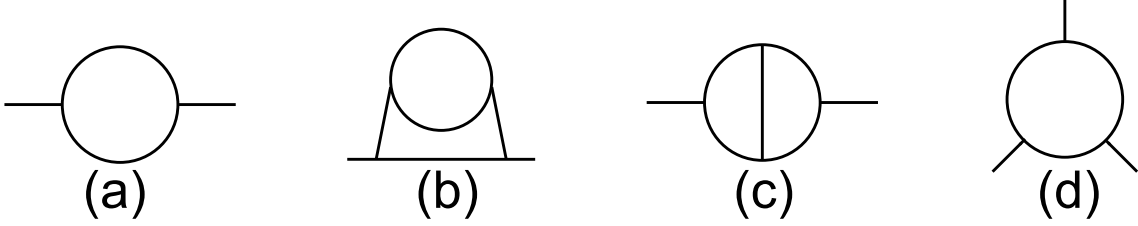


Figure 1: Divergent diagrams.

### 3 Regularization of the theory

Let us assume that some UV regularization of the theory is introduced so that all diagrams are UV finite. As mentioned in the Introduction, the results of calculations of diagrams in LF perturbation theory and usual covariant perturbation theory in Lorentz coordinates can differ. To find these differences one can apply the method of [4, 12, 14] if the regularization  $|k_-| \geq \varepsilon > 0$  is used. As was noted in [12] for the theory with the propagator containing additional pole in  $k_-$  (like in eq. (4)), these differences arise for diagrams with any number of external lines shown in Fig. 2(a). As the compensation of these differences could require the addition of infinite number of counterterms to the action, we need some modification of the propagator that removes these differences. A way to do this simultaneously with the introduction of UV regularization was proposed in [12], and we will use its analog.

The main idea of this method is the introduction of gauge field analog of PV ghost fields with the simultaneous introduction of higher (noncovariant) derivatives (that breaks gauge invariance). Let us note that, owing to the coordinates of the light front, an introduction of higher derivative in the form of the power one of  $\partial^2$  does not lead to a complication of canonical formalism on the LF. This is because the action can be transformed by integration by parts to the form containing no higher than the first derivatives in  $x^+$ . So we limit ourselves by just those higher derivatives. Let us choose the Lagrangian density in the form

$$\mathcal{L} = \sum_{j=0}^2 \left( -\frac{1}{4} f_{j,\mu\nu}^a \left( \frac{M_j^2 + \partial_\parallel^2}{B_j} \right) f_{j,\mu\nu}^a + \frac{m}{2} \varepsilon^{\mu\nu\alpha} A_{j,\mu}^a \left( \frac{M_j^2 + \partial_\parallel^2}{B_j} \right) \partial_\nu A_{j,\alpha}^a \right) + g f^{abc} A_\mu^a A_\nu^b \partial^\mu A^{c\nu}. \quad (5)$$

Here  $f_{j,\mu\nu}^a = \partial_\mu A_{j,\nu}^a - \partial_\nu A_{j,\mu}^a$ , the quantity  $A_{0,\mu}^a$  is the physical gluon field, and  $A_{1,\mu}^a$  and  $A_{2,\mu}^a$  are extra fields, PV fields analog. As one can see, the quadratic part of the Lagrangian

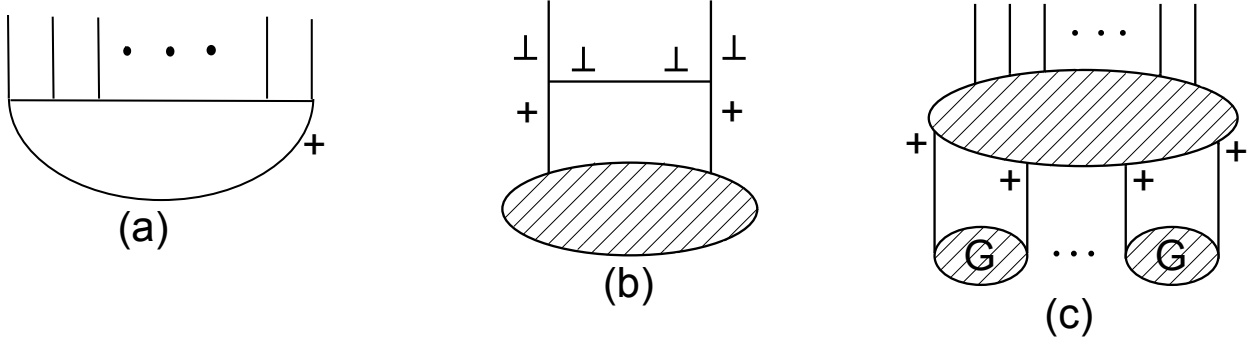


Figure 2: (a) The example of diagram which has different values in perturbation theory on the LF and usual covariant perturbation theory. (b) General form of diagram having mentioned above different values for the theory with single PV field. (c) General form of diagram which can contain IR divergency. Symbols  $+$ ,  $\perp$  denote indices of propagators, hatched domains in diagrams denote arbitrary subdiagrams.

is diagonal in these fields, and only their sum  $A_\mu^a = A_{0,\mu}^a + A_{1,\mu}^a + A_{2,\mu}^a$  enters into the interaction term. The conditions  $A_{j,-}^a = 0$  for physical and extra fields, analogous to the LF gauge, are proposed. Let us remark that the parameter  $m$ , giving the mass to the gauge field, is common for all three fields  $A_{j,\mu}^a$ , and the differences between them are related to the values of parameters  $M_j$ ,  $B_j$  (for conventional PV fields parameters analogous to  $M_j$  correspond to masses of fields).

As the interaction contains the sum of fields  $A_\mu^a$ , propagators of three fields sum up in diagrams, and Feynman integrals can be written in terms of summarized propagator  $\Delta_{\mu\nu}^{ab}$ , i.e. the sum of individual field propagators

$$\Delta_{j,\mu\nu}^{ab} = \frac{i\delta^{ab}B_j}{(k^2 - m^2 + i0)(k_\parallel^2 - M_j^2 + i0)} \left( g_{\mu\nu} - \frac{k_\mu n_\nu + n_\mu k_\nu + i m \varepsilon_{\mu\nu\alpha} n^\alpha}{k_\parallel^2 + i0} 2k_+ \right). \quad (6)$$

We relate the parameters  $M_0$ ,  $M_1$  and  $M_2 \equiv \mu$  to regularization parameters and choose the quantities  $B_j$  so that to assure the decrease of summarized propagator as  $1/k_\parallel^6$  (see the discussion of the necessity of such exceeded requirement in the next Sect.). On the other hand, we choose them so that to cancel the additional pole in  $k_-$ , present in the propagator. This can be done if one takes

$$B_0 = \frac{M_0^4 M_1^2}{(M_1^2 - M_0^2)(M_0^2 - \mu^2)}, \quad B_1 = -\frac{M_0^2 M_1^4}{(M_1^2 - M_0^2)(M_1^2 - \mu^2)},$$

$$B_2 = -\frac{M_0^2 M_1^2 \mu^2}{(M_0^2 - \mu^2)(M_1^2 - \mu^2)}, \quad (7)$$

resulting in the following form of the regularized summarized propagator:

$$\Delta_{\mu\nu}^{ab} = \frac{-i\delta^{ab}}{(k^2 - m^2 + i0) \left(1 - \frac{k_{\parallel}^2 + i0}{M_0^2}\right) \left(1 - \frac{k_{\parallel}^2 + i0}{M_1^2}\right)} \times \frac{g_{\mu\nu}k_{\parallel}^2 - (k_{\mu}n_{\nu} + k_{\nu}n_{\mu} + i m \varepsilon_{\mu\nu\alpha}n^{\alpha})2k_{+}}{k_{\parallel}^2 - \mu^2 + i0}. \quad (8)$$

It is easy to check that all diagrams of Fig. 1 become finite with that propagator. After removing of the regularization the propagator (8) must turn into the propagator of non-regularized theory (4). It is easy to see that the following conditions must be fulfilled when one removes the regularization:

$$\mu \rightarrow 0, \quad M_0 \rightarrow \infty, \quad M_1 \rightarrow \infty, \quad \frac{M_1}{M_0} \rightarrow \infty. \quad (9)$$

The latter one is necessary as we want that only physical field  $A_{0,\mu}^a$  remains in the regularization removing limit, because the propagator (6) of this field turns under that condition into the expression (4), and propagators of the other two fields tend to zero.

## 4 Comparison of perturbation theory on the LF and covariant one in Lorentz coordinates

Using the already mentioned above method proposed in [14] it is possible to analyse the difference between the results of calculations of diagrams in LF perturbation theory and the usual covariant perturbation theory in Lorentz coordinates when the regularization  $|k_{-}| \geq \varepsilon$  is applied. It is possible to calculate this difference as the difference between the diagram for which the integration is over all momenta  $k_{-}$  and the same diagram for which that integration is only over the domain  $|k_{-}| \geq \varepsilon > 0$  (here  $k_{-}$  is the propagator momentum). Thus the difference is the sum of all copies of the diagram, in which one integrates over the momentum  $k_{-}$  over the domain  $|k_{-}| \geq \varepsilon > 0$  at least for one of propagator momenta. The idea of the method is the following. If one makes for each loop momentum  $k$  (which can be always identified with some propagator momentum) the change  $k_{-} \rightarrow k_{-}\varepsilon$ ,  $k_{+} \rightarrow k_{+}/\varepsilon$ , an essential dependence on  $\varepsilon$  in the integration region disappears, and one can investigate the behavior of the integrand for an arbitrarily complicated diagram.

In result, the above mentioned difference for an arbitrary diagram can be estimated in every contribution to it in the form of  $\varepsilon^{\sigma}$  where  $\sigma$  is determined by topology of the diagram, its Lorentz structure and general properties of the theory such as spin of the field and UV properties of the propagators (see details in [14]).

For example let us consider the diagram shown in Fig. 1(a). Let  $p_{\mu}$  denotes external momentum and  $k_{\mu}$  denotes the loop momentum coinciding with one of propagator momenta. In perturbation theory on the LF this momentum is limited by the condition  $|k_{-}| \geq \varepsilon$  and, when one of the differences for that diagram is calculated, it is limited by the condition  $|k_{-}| \leq \varepsilon$ . Let us write such contribution to the difference, when the above

mentioned momentum  $k$  corresponds to the component  $\Delta_{++}$  of the propagator:

$$\begin{aligned}
& \int_{-\infty}^{\infty} dk_{\perp} \int_{-\infty}^{\infty} dk_{+} \int_{-\varepsilon}^{\varepsilon} dk_{-} \frac{(2p_{-} - k_{-})^2 (2k_{+})^2}{(k^2 - m^2 + i0) (k_{\parallel}^2 - \mu^2 + i0)} \times \\
& \quad \times \frac{2(p_{+} - k_{+})(p_{-} - k_{-}) M_0^4 M_1^4}{(k_{\parallel}^2 - M_0^2 + i0) (k_{\parallel}^2 - M_1^2 + i0) ((p - k)^2 - m^2 + i0)} \times \\
& \quad \times \frac{1}{(2(p_{+} - k_{+})(p_{-} - k_{-}) - \mu^2 + i0) (2(p_{+} - k_{+})(p_{-} - k_{-}) - M_0^2 + i0)} \times \\
& \quad \times \frac{1}{(2(p_{+} - k_{+})(p_{-} - k_{-}) - M_1^2 + i0)}. \quad (10)
\end{aligned}$$

Here the first factor  $(2p_{-} - k_{-})^2$  in the numerator of the integrand corresponds to vertices of the diagram. After the change  $k_{-} \rightarrow k_{-}\varepsilon$ ,  $k_{+} \rightarrow k_{+}/\varepsilon$  this integral takes the following form:

$$\begin{aligned}
& \int_{-\infty}^{\infty} dk_{\perp} \int_{-\infty}^{\infty} dk_{+} \int_{-1}^1 dk_{-} \frac{(2p_{-} - \varepsilon k_{-})^2 (2k_{+})^2}{\varepsilon^2 (k^2 - m^2 + i0) (k_{\parallel}^2 - \mu^2 + i0)} \times \\
& \quad \times \frac{2(p_{+} - k_{+}/\varepsilon)(p_{-} - \varepsilon k_{-}) M_0^4 M_1^4}{(k_{\parallel}^2 - M_0^2 + i0) (k_{\parallel}^2 - M_1^2 + i0) (2(p_{+} - k_{+}/\varepsilon)(p_{-} - \varepsilon k_{-}) - (p_{\perp} - k_{\perp})^2 - m^2 + i0)} \times \\
& \quad \times \frac{1}{(2(p_{+} - k_{+}/\varepsilon)(p_{-} - \varepsilon k_{-}) - \mu^2 + i0) (2(p_{+} - k_{+}/\varepsilon)(p_{-} - \varepsilon k_{-}) - M_0^2 + i0)} \times \\
& \quad \times \frac{1}{(2(p_{+} - k_{+}/\varepsilon)(p_{-} - \varepsilon k_{-}) - M_1^2 + i0)}. \quad (11)
\end{aligned}$$

In the limit  $\varepsilon \rightarrow 0$  this integral equals to the following expression:

$$\begin{aligned}
& \int_{-\infty}^{\infty} dk_{\perp} \int_{-\infty}^{\infty} dk_{+} \int_{-1}^1 dk_{-} \frac{\varepsilon (2p_{-})^2 (2k_{+})^2}{(k^2 - m^2 + i0) (k_{\parallel}^2 - \mu^2 + i0)} \times \\
& \quad \times \frac{2(-k_{+})p_{-} M_0^4 M_1^4}{(k_{\parallel}^2 - M_0^2 + i0) (k_{\parallel}^2 - M_1^2 + i0)} \frac{1}{(2(-k_{+})p_{-} + i0)^4} = \\
& = -2\varepsilon M_0^4 M_1^4 \int_{-\infty}^{\infty} dk_{\perp} \int_{-\infty}^{\infty} dk_{+} \int_{-1}^1 dk_{-} \frac{1}{(k_{+}p_{-} - i0) (k^2 - m^2 + i0)} \times \\
& \quad \times \frac{1}{(k_{\parallel}^2 - \mu^2 + i0) (k_{\parallel}^2 - M_0^2 + i0) (k_{\parallel}^2 - M_1^2 + i0)}. \quad (12)
\end{aligned}$$

This expression tends to zero in the limit  $\varepsilon \rightarrow 0$  at fixed parameters  $\mu, M_{0,1}$ . This determines the order of the regularizations removing: firstly  $\varepsilon \rightarrow 0$ , then  $\mu \rightarrow 0$ ,  $M_{0,1} \rightarrow \infty$ , taking into account (9).

We have shown how one of contributions to the difference disappears in the limit  $\varepsilon \rightarrow 0$  for the diagram in Fig. 1(a). Following the method proposed in [14] one can succeed in showing that the same is true for all possible contributions to the differences for any diagrams of the considered theory in any order of perturbation theory. Let us remark that



in considered theory the diagrams with all external lines, joined to single vertex, are absent (diagrams with one external line are equal to zero due to the global SU(N) symmetry and the 1-particle irreducible diagrams with two external lines, joined to single vertex, are absent due to the absence of vertices with four lines). If such diagrams existed in the theory they could give a difference between LF perturbation theory and usual covariant perturbation theory (in calculations on the LF such diagrams are equal to zero but they can be nonzero in usual covariant perturbation theory, see [14]).

It may be remarked that the ultimate absence of differences for all diagrams is owing to sufficiently fast decrease of the propagator (as  $1/k_{\parallel}^6$ ). It is easy to note that it would be sufficient to have the decrease as  $1/k_{\parallel}^4$  for the UV finiteness. This can be done by introducing not two PV fields (as have been made in Sect. 3) but only a single one. That could simplify the theory. However in this case the differences between calculation on the LF and in the usual covariant perturbation theory in the limit  $\varepsilon \rightarrow 0$  disappears not for all diagrams. The finite in that limit differences would be nonzero for infinite number of diagrams having the form shown in Fig. 1(b). These differences turn out to be divergent in the UV regularization removing limit. To compensate them it would be necessary to add to the LF Hamiltonian some new (having the gluon mass form) counterterm with UV divergent coefficient (being the sum of contributions of infinite number of differences). We note here that one can fully avoid the appearance of unknown coefficients before the counterterms at UV renormalization in (2+1)-dimensions (see below). So it is reasonable to choose a variant of the theory in which they do not appear also due to comparison of perturbation theory on the LF and the usual covariant perturbation theory. That is what we do in the present paper.

## 5 Analysis of longitudinal IR divergences

It was shown in the previous section that, if we use the introduced above analog of PV regularization, the diagrams of perturbation theory on the LF transform into the diagrams of the usual covariant perturbation theory in the limit  $\varepsilon \rightarrow 0$ . In the next section we show that these diagrams can be renormalized in such a way that they coincide with corresponding diagrams in dimensional regularization (and renormalization) in the regularization removing limit (9). Furthermore it is possible to go to the Euclidean form of the theory by Wick rotation because with the Mandelstam-Leibbrandt prescription the structure of poles allows to do that. After the Wick rotation propagator (8) takes the form

$$\Delta_{\mu\nu}^{ab} = \frac{-i\delta^{ab}}{(k^2 + m^2) \left(1 + \frac{k_{\parallel}^2}{M_0^2}\right) \left(1 + \frac{k_{\parallel}^2}{M_1^2}\right)} \times \frac{\delta_{\mu\nu}k_{\parallel}^2 - (k_{\mu}n_{\nu} + n_{\mu}k_{\nu} - m\varepsilon_{\mu\nu\alpha}n_{\alpha})2k_{\beta}n_{\beta}^*}{k_{\parallel}^2 + \mu^2}. \quad (13)$$

Here the vector  $n_{\mu}$  becomes complex vector with components  $n_0 = -\frac{i}{\sqrt{2}}$ ,  $n_1 = \frac{1}{\sqrt{2}}$ ,  $n_{\perp} = 0$ , and the vector  $n_{\beta}^*$  is the result of its complex conjugation. Let us note that, despite of the transition to Euclidean space, it is possible to use the indices  $-$  and  $+$  as before implying by them the contraction with vectors  $n_{\mu}$  and  $n_{\mu}^*$ , respectively. Taking into account the



decomposition  $\delta_{\mu\nu} = n_\mu n_\nu^* + n_\mu^* n_\nu + \delta_{\mu\perp} \delta_{\nu\perp}$  it follows that in Euclidean space one can write  $a_\mu b_\mu = a_+ b_- + a_- b_+ + a_\perp b_\perp$ .

Further we analyse the limit  $\mu \rightarrow 0$  for arbitrary diagram. As it is seen from the form of the propagator (13) the essential (i.e. appearing at any values of external momenta) IR divergences can appear in this limit at the points of the momentum space at which the quantities  $k_\parallel^2 = k_1^2 + k_2^2$  become equal to zero for several propagator momenta simultaneously. Note that every propagator gives the pole of the first order in  $k_\parallel$ , and only for the component  $\Delta_{+\perp}$  (let us also note that  $\Delta_{-\nu} = 0$  and that here and further we discard color indices). In the paper [12] the analysis was carried out of the possibility of the appearance of the longitudinal IR divergency for the (3+1)-dimensional QCD with the analogous regularization, when the gluon propagator has the same properties. Repeating this analysis for the now considered model it is possible to find that the above mentioned divergence can be only logarithmic, and it can appear only for diagrams of the type shown in Fig. 2(c), and only for contributions of the form

$$\Delta_{+\nu} G_{\nu\gamma} \Delta_{\gamma+} = n_\mu^* \Delta_{\mu\nu} G_{\nu\gamma} \Delta_{\gamma\delta} n_\delta^*, \quad (14)$$

where  $G_{\nu\gamma}$  is one of marked in Fig. 2(c) subdiagrams with two external lines (not necessarily 1-particle irreducible in general).

Let us analyse the contribution of the expression (14) which gives the longitudinal IR divergence. First we write down the contribution of the pole in  $k_\parallel$  for one of the quantities  $\Delta_{\mu\nu}$  entering into (14) keeping only essential terms in which the cancelation of the pole does not take place, and also discarding nonessential total factor:

$$n_\mu^* \frac{\delta_{\mu\nu} k_\parallel^2 - (k_\mu n_\nu + n_\mu k_\nu - m \varepsilon_{\mu\nu\alpha} n_\alpha) 2k_\beta n_\beta^*}{k_\parallel^2} G_{\nu\gamma} \rightarrow -\frac{2k_\beta n_\beta^*}{k_\parallel^2} (k_\nu - m n_\mu^* \varepsilon_{\mu\nu\alpha} n_\alpha) G_{\nu\gamma}. \quad (15)$$

Let us note that the vector  $n_\mu^* \varepsilon_{\mu\nu\alpha} n_\alpha$  has only transversal component. At the analysis of IR divergency we can suppose that  $k_\perp \neq 0$  (the integration over all momenta at  $k_\perp = 0$  and  $k_\parallel = 0$  does not lead to IR divergency while we have logarithmic IR divergency in  $k_\parallel$ ). Then the mentioned above constant vector can be written in the form

$$n_\mu^* \varepsilon_{\mu\nu\alpha} n_\alpha = -i \frac{k_\nu - k_0 \delta_{\nu 0} - k_1 \delta_{\nu 1}}{k_\perp}. \quad (16)$$

After that the essential part (15) can be written, again discarding the terms in which the cancelation of the pole takes place, in the form

$$-\frac{2k_\beta n_\beta^*}{k_\parallel^2} \left(1 + i \frac{m}{k_\perp}\right) k_\nu G_{\nu\gamma}. \quad (17)$$

If gauge invariance was conserved this expression would be equal to zero as a consequence of the Ward identities, and hence the essential longitudinal divergences would be absent in this case. Exactly the same result was obtained in [12] for (3+1)-dimensional QCD. Thus the result does not change when we take into account the influence of the CS term. However the used regularization violates gauge invariance, regularizing the emerging divergence by the parameter  $\mu$ . To avoid the divergence in the limit  $\mu \rightarrow 0$  it is necessary that simultaneously with taking this limit the UV regularization be removed (i.e. all

limits be taken simultaneously (9)) and renormalizing counterterms be chosen so that in the regularization removing limit Ward identities be satisfied (the idea of that mechanism was supposed in [12]). With the choice of counterterms in such a way that the values of diagrams differ from renormalized results, obtained via dimensional regularization, by the amount of the order  $\frac{1}{M_0}$  (for UV finite diagrams this is automatically true if the product  $\mu M_0$  is bounded from above) the contribution of the quantity (17)) can be estimated as  $O\left(\frac{1}{M_0}\right)$ . Therefore the corresponding to it total contribution to the diagram (which is equal to zero in dimensional regularization) can be estimated as  $\frac{(\ln \mu)^N}{M_0}$  (where  $N$  is the number of subdiagrams  $G$  in Fig. 2(c)). Let us require that this relation tends to zero for any  $N$ . Then diagrams for which the longitudinal divergency can appear will not give, in the regularization removing limit, differences between results of calculations in the scheme used here and those in the dimensional regularization scheme.

So one can conclude that if one takes, for example,

$$\mu \sim \frac{1}{\Lambda}, \quad M_0 \sim \Lambda, \quad M_1 \sim \Lambda^2, \quad (18)$$

all required conditions (including the conditions (9)) are satisfied in the limit  $\Lambda \rightarrow \infty$ , and one can take  $\mu = 0$  in diagrams for the analysis of the UV divergences.

## 6 Renormalization of the theory

In the considered theory we have to renormalize only finite number of diagrams shown in Fig. 1. We can calculate their values in the used regularization and therefore find explicitly the coefficients of the counterterms. This provides the coincidence of the values for these diagrams with the results of their renormalization obtained in the dimensional regularization. In the result, inspite of the violation of Lorentz and gauge symmetries in the used regularization, these symmetries are restored for the renormalized theory in the regularization removing limit  $\Lambda \rightarrow \infty$ . Further we choose the counterterms of our theory so that the renormalized diagrams in our regularization coincide with renormalized diagrams in dimensional regularization.

Let us consider the diagram shown in Fig. 1(a). After the regularization for that diagram we consider its Taylor decomposition in the external momentum  $p_\mu$  in the vicinity of the point  $p_\mu = 0$ . Using dimensional analysis of its UV divergent parts one can find that for the linearly divergent diagram it is sufficient to renormalize only the first two terms in this decomposition, and only the first term for the logarithmically divergent diagrams. In the used regularization this diagram contains integrals that equal zero when the external upper indices are  $++$ ,  $+\perp$  and  $\perp+$ . One of two reasons can explain this. The first one is the odd parity of integrand with respect to the one of momentum components. The second reason is a possibility to express the integrand as the difference of two parts that cancel each other due to the symmetry under the interchange of longitudinal components of the integration momentum. Note that we don't consider amputated diagrams with the upper index  $-$ , because in the  $A_- = 0$  gauge they don't contribute to corresponding Green functions due to contractions with propagators. For the indices  $\perp\perp$  we have the Euclidean

form of the integral (for  $p_\mu = 0$ )

$$\int_{-\infty}^{\infty} dk_{\perp} \int_{-\infty}^{\infty} dk_0 \int_{-\infty}^{\infty} dk_1 \frac{(k_{\perp}^2 - k_0^2 - k_1^2)}{(k^2 + m^2)^2} R(k_0, k_1, M_0, M_1),$$

$$R(k_0, k_1, M_0, M_1) = \frac{M_0^4 M_1^4}{\left(k_{\parallel}^2 + M_0^2\right)^2 \left(k_{\parallel}^2 + M_1^2\right)^2}. \quad (19)$$

In this integral we set  $\mu = 0$ , because it is IR-finite. Using cylindrical coordinates ( $\varphi$ ,  $\rho = \sqrt{k_0^2 + k_1^2}$ ,  $k_{\perp}$ ) one can perform the integration over the angle variable that gives the factor  $2\pi$ :

$$\begin{aligned} \pi \int_0^{\infty} d\rho \int_{-\infty}^{\infty} dk_{\perp} \frac{(k_{\perp}^2 - \rho) R(\rho, M_0, M_1)}{(\rho + k_{\perp}^2 + m^2)^2} &= \\ &= \frac{\pi^2}{2} \int_0^{\infty} d\rho \left( \frac{1}{(\rho + m^2)^{\frac{1}{2}}} - \frac{\rho}{(\rho + m^2)^{\frac{3}{2}}} \right) R(\rho, M_0, M_1) = \\ &= \frac{\pi^2 m^2}{2} \int_0^{\infty} d\rho \frac{R(\rho, M_0, M_1)}{(\rho + m^2)^{\frac{3}{2}}}. \end{aligned} \quad (20)$$

Now we can remove the regularization ( $M_{0,1} \rightarrow \infty$  and correspondingly  $R \rightarrow 1$ ) and compute the integral

$$\frac{\pi^2 m^2}{2} \int_{m^2}^{\infty} \frac{d\rho}{(\rho + m^2)^{\frac{3}{2}}} = \frac{\pi^2 m^2}{2} \int_{m^2}^{\infty} \frac{d\rho}{\rho^{\frac{3}{2}}} = \pi^2 m. \quad (21)$$

Thus one can see that the linear divergence, in fact, is absent. We get the same answer using dimensional regularization. We have computed the second term in the Taylor series with an analytical computer program and found that this term equals zero for all concerned external indices ( $++$ ,  $+\perp$ ,  $\perp+$  and  $\perp\perp$ ) of the diagram. This means that the diagram in Fig. 1(a) needs no renormalization. Analogously, using an analytical computer program we found that the UV divergent part of the diagram shown in Fig. 1(d) equals zero in both regularizations and for all concerned external indices ( $+++$ ,  $++\perp$ ,  $\dots$ ). We have not found an analytical answer for the divergent in the limit  $M_{0,1} \rightarrow \infty$  parts of the remaining two diagrams. However, these divergent parts can be calculated numerically and this is sufficient for possible non-perturbative computations involving the LF Hamiltonian.

In that way we have demonstrated the possibility to exactly find the counterterms that are needed for renormalization. Thus now we can construct the renormalized Hamiltonian [4, 12] and use it for non-perturbative computations.

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